IGMO Christmas Edition Solutions

Note:

This was an unofficial version released by the team. Hence there are no submission statistics. However, we did encourage users from the discord server affiliated with IGMO (IΓMC) to send their solutions, and wherever an appropriate solution was received, the user who submitted it was mentioned. If the official solution was vastly different from the submitted solution, it was also included. The affiliated discord server is linked on the website

Problem 1 :

Santa Claus decorates his Christmas tree with a decoration which has a shape of a regular 12-sided polygon. Let the 12-sided polygon be $A_1A_2A_3...A_{12}$. Suppose I_1, I_2 and I_3 are the incentres of $\triangle A_1A_2A_5$, $\triangle A_5A_7A_8$ and $\triangle A_8A_{11}A_1$ respectively. Prove that I_1A_8, I_2A_1 and I_3A_5 are concurrent.

Solution

Problem 2 :

Santa has almost finished decorating his giant gingerbread house for Christmas. The only thing left to do is to create a circular fence around it. For this purpose Santa wants to use $n \geq 2$ candy canes in 3 colors: Green, Red and White, but he doesn't want any two adjacent candy canes to have the same color. Find the number of possible arrangements of this fence in terms of n
Note : Single candy canes are distinguishable.

Solution

Problem 3 :

For $n \geq 2$, let $a_1, a_2, \ldots, a_n$ be reals such that $a_1 + a_2 + \cdots + a_n = n^n - 1$.

Show that

$$a_1^2 + \frac{a_2^2}{1+a_1^2} + \cdots + \frac{a_n^2}{1+a_1^2+a_2^2+\cdots+a_{n-1}^2} > n\left(\frac{n^2}{\sqrt[n]{n+1}} - 1\right)$$

Solution

Problem 4 :

Let f and g be real-valued functions defined for all real numbers x and a, and s, m be some positive constants, such that f, g satisfy the equations

$$f(x+a)+f(x-a)=\frac{2f(x)g(a)}{s}, \quad |f(x)| \leq m$$

for all x, a. Prove that if $|f|$ is not identically zero, and attains a maximum value, then $|g(a)| \leq s$ for all a.

Solution

Problem 5 :

There are some (at least 3) elves in Santa's backyard. The backyard has a circular shape with diameter d. Santa finds that any three elves can be surrounded by an $\ell \times d$ rectangle. Prove that all the elves can be surrounded by a $2\ell \times d$ rectangle.

Note: An elf being "surrounded" by a rectangle means that the point corresponding to the elf is contained within the rectangle, or is on its perimeter. The elves have no areas, they are points.

Solution

Problem 6 :

For Christmas, Santa gifts us a special machine. This special machine takes as input any relatively prime positive integers a, n and returns the order of a modulo n as the output, that is to say : returns the least positive integer b such that $a^b \equiv 1 \mod n$. Using this special machine, devise an algorithm of time complexity at most $O(\log(n))$ to factorize natural numbers n of the form pq, where p, q are safe primes(which means p,q,$\frac{p-1}{2}$,$\frac{q-1}{2}$ are all primes greater than 5).

Note : You should assume that calling the special machine is $O(1)$, and for two positve integers a and b, calculating $\gcd(a, b)$ is $O(\log(\max(a, b))$.

Solution

Solution to P1 :

Submitted by `@getthezucc#1175` on discord:
By Trigonometric Ceva's Theorem, A_8I_1, A_1I_2, A_5I_3 are concurrent if

$$\frac{\sin(\angle A_5A_1I_2)}{\sin(\angle A_8A_1I_2)}\frac{\sin(\angle A_1A_8I_1)}{\sin(\angle A_5A_8I_1)}\frac{\sin(\angle A_8A_5I_3)}{\sin(\angle A_1A_5I_3)}=1$$

Note that

$$\frac{\sin(\angle A_8A_1I_2)}{\sin(\angle A_5A_1I_2)}=\frac{d(I_2,A_1A_8)}{d(I_2,A_1A_5)}=\frac{I_2A_8\cos(\angle I_2A_8A_7)}{I_2A_5\cos(\angle I_2A_5A_7)}$$

$$=\frac{\sin(\angle I_2A_5A_8)}{\sin(\angle I_2A_8A_5)}\frac{\cos(\angle I_2A_8A_7)}{\cos(\angle I_2A_5A_7)}$$

Where sine rule is applied on the second line. Similarly,

$$\frac{\sin(\angle A_5A_8I_1)}{\sin(\angle A_1A_8I_1)}=\frac{\sin(\angle I_1A_1A_5)}{\sin(\angle I_2A_5A_1)}\frac{\cos(\angle I_2A_5A_2)}{\cos(\angle I_2A_1A_2)}$$

$$\frac{\sin(\angle A_1A_5I_3)}{\sin(\angle A_8A_5I_3)}=\frac{\sin(\angle I_3A_8A_1)}{\sin(\angle I_3A_1A_8)}\frac{\cos(\angle I_3A_1A_{11})}{\cos(\angle I_3A_8A_{11})}$$

Multiplying everything,

$$\frac{\sin(\angle I_2A_5A_8)}{\sin(\angle I_2A_8A_5)}\frac{\cos(\angle I_2A_8A_7)}{\cos(\angle I_2A_5A_7)}\frac{\sin(\angle I_1A_1A_5)}{\sin(\angle I_2A_5A_1)}\frac{\cos(\angle I_2A_5A_2)}{\cos(\angle I_2A_1A_2)}\frac{\sin(\angle I_3A_8A_1)}{\sin(\angle I_3A_1A_8)}\frac{\cos(\angle I_3A_1A_{11})}{\cos(\angle I_3A_8A_{11})}$$

$$=\frac{\sin(\frac{\pi}{24})}{\sin(\frac{\pi}{12})}\frac{\cos(\frac{\pi}{12})}{\cos(\frac{\pi}{24})}\frac{\sin(\frac{\pi}{8})}{\sin(\frac{\pi}{24})}\frac{\cos(\frac{\pi}{24})}{\cos(\frac{\pi}{8})}\frac{\sin(\frac{\pi}{12})}{\sin(\frac{\pi}{8})}\frac{\cos(\frac{\pi}{8})}{\cos(\frac{\pi}{12})}=1$$

Official Solution :

$$\angle A_1A_5A_2=\angle A_7A_5A_8=\frac{360^o}{12}\times\frac{1}{2}=15^0$$

$$\angle I_1A_5A_1=\angle I_2A_5A_8=\frac{15^o}{2}=7.5^0$$

$$\angle A_5A_8A_7 = \angle A_1A_8A_{11} = \frac{360^o}{12} \times 2 \times \frac{1}{2} = 30^0$$
$$\angle I_2A_8A_5 = \angle I_3A_8A_1 = \frac{30^o}{2} = 15^0$$
$$\angle A_2A_1A_5 = \angle A_8A_1A_{11} = \frac{360^o}{12} \times 3 \times \frac{1}{2} = 45^0$$
$$\angle I_1A_1A_5 = \angle I_3A_1A_8 = \frac{45^o}{2} = 22.5^0$$

Consider $\triangle I_1A_1A_5$, $\triangle I_2A_5A_8$ and $\triangle I_3A_8A_1$, since $\angle I_1A_5A_1 = \angle I_2A_5A_8$, $\angle I_2A_8A_5 = \angle I_3A_8A_1$, $\angle I_1A_1A_5 = \angle I_3A_1A_8$, by Jacobi's theorem, I_1A_8, I_2A_1 and I_3A_5 are concurrent.

Solution to P2 :

Submitted by `@getthezucc#1175` on discord:

Let the colors be represented by $1, \omega, \omega^2$ where ω is the 3^{rd} root of unity. The condition that every adjacent colour needs to be distinct can be represented by
$$(\omega + \omega^2)^{n-1} = a + b \cdot \omega + c \cdot \omega^2$$
where
$$a + b + c = 2^{n-1}$$
By symmetry $b = c$ (substituting ω with ω^2 also works). Furthermore, since $\omega + \omega^2 = -1$, the equations can be simplified to
$$a - b = (-1)^{n-1} \Rightarrow b = \frac{2^{n-1} - (-1)^{n-1}}{3}$$
An arrangement corresponds to either b or c, since the last colour needs to be different from the first, where the first colour can be chosen arbitrarily, the answer is hence
$$6b = 2(2^{n-1} - (-1)^{n-1})$$
Thus, the number of possible arrangements in terms of n is
$$2^n + 2(-1)^n$$

Alternate Solution submitted by `@MountainC#8098` on discord:

We can label each fence post around the circle from 1 to n, giving a bijection with sequences of n posts where adjacent colours are different (condition 1), with the extra condition that post 1 and post n also need to be different (2) since the posts lie in a circle.
Ignore (2) for the moment and consider only (1). After choosing a colour for the first post (3 choices), each post after has 2 choices, giving $g(n) = 3 \cdot 2^{n-1}$ sequences satisfying (1).
We now find the proportion of sequences satisfying (2) via recursion. Let $r(n)$ denote this number, so that our final answer is $f(n) = g(n)r(n)$. Then there are two possibilities for the second last colour ($n \geq 2$): if it is the same as the first (with ratio $1 - r(n-1)$), then (2) is guaranteed, and if it is different then (2) only holds for $\frac{1}{2}$ of these sequences by symmetry (each sequence as such has a partner with the other two colours swapped). We thus obtain the recursive formula

$$r(n) = (1 - r(n-1)) \cdot 1 + r(n-1) \cdot \frac{1}{2}, \quad (n \geq 2)$$

which can be solved with $r(2) = 1$ to give

$$\begin{aligned} r(n) &= 1 - \frac{1}{2} + \frac{1}{4} - \cdots + (-1)^{n-2}\frac{1}{2^{n-2}} \\ &= \frac{1 - (-1)^{n-1}\frac{1}{2^{n-1}}}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{1}{3}\left(2 + \frac{(-1)^n}{2^{n-2}}\right), \end{aligned}$$

which gives the final answer

$$\begin{aligned} f(n) &= g(n)r(n) \\ &= 3 \cdot 2^{n-1} \cdot \frac{1}{3}\left(2 + \frac{(-1)^n}{2^{n-2}}\right) \\ &= \boxed{2^n + 2(-1)^n}. \end{aligned}$$

Appendix
To clarify, the note means shifting an arrangement cyclically around the fence counts as a new arrangement (if individual positions of the fence have different colours). Removing this condition, the problem can still be solved easily for prime n (divide by n) but another approach will be needed for composite numbers.

Solution to P3 :

Solution submitted by `@Bernat#9585` on discord : Say $a_0 = 1$. Adding n to both sides of the inequality, the problem becomes equivalent to

$$\sum_{i=0}^{n-1} \frac{\sum_{j=0}^{i+1} a_j^2}{\sum_{j=0}^{i} a_j^2} > n\left(\frac{n^{2n}}{n+1}\right)^{\frac{1}{n}}$$

The almost telescoping fractions motivate direct AM-GM:

$$\sum_{i=0}^{n-1} \frac{\sum_{j=0}^{i+1} a_j^2}{\sum_{j=0}^{i} a_j^2} \geq n(\sum_{i=0}^{n} {a_i}^2)^{\frac{1}{n}}$$

Then, a straightforward QM-AM gives:

$$\sum_{i=0}^{n} {a_i}^2 > \frac{\left(\sum_{i=0}^{n} a_i\right)^2}{n+1} = \frac{n^{2n}}{n+1}$$

Joining both inequalities yields our desired result. Note that the inequality is strict since all a_i cannot be equal.

Solution to P4 :

Submitted by `@MountainC#8098` on discord:

Suppose $|f|$ has a maximum at $x = \beta$. Noting that $|f(\beta)| > 0$ since $|f|$

is not identically zero,

$$\begin{aligned}
\frac{2f(\beta)g(a)}{s} &= f(\beta+a)+f(\beta-a), \\
\left|\frac{2f(\beta)g(a)}{s}\right| &= \left|f(\beta+a)+f(\beta-a)\right|, && \text{(abs. value)} \\
2|f(\beta)|\frac{|g(a)|}{s} &\leq |f(\beta+a)|+|f(\beta-a)|, && \text{(triangle inequality)} \\
2|f(\beta)|\frac{|g(a)|}{s} &\leq 2|f(\beta)|, && (|f| \text{ max. at } \beta) \\
\frac{|g(a)|}{s} &\leq 1, && (|f(\beta)|>0)
\end{aligned}$$

which is as desired.

Solution to P5 :

Solution submitted by `@Bernat#9585` on discord

When $l \geq d$, the rectangle covers the whole circle so the result is trivially true. When $l < d$, since the maximal distance from two points is d, the rectangle in the problem is equivalent to an infinitely long strip made of two paralel lines in which the distance from each other is l. Consider the two furthest points, and then the furthest point from the line passing through those two points. We claim that the optimal length of the strip is precisely the distance between the 3rd point and the line. Suppose that in an optimal strip the two parallel lines intersect at only 2 points, one each. Then by rotating the parallel lines with centres each point in the correct direction until the 3rd point touches one of the lines, the distance between the two lines will decrease. Therefore, in the optimal strip, all 3 points lie on the parallel lines. Using that $A = \frac{1}{2}hb$, the height is minimized by maximizing its base. Thus, the optimal will be to pick the two furthest points to be in the first line and the 3rd point in the second one. This gives, then, a strip of length l, as desired. Drawing the final strip of length $2l$ as the union of this strip and the symmetric with respect to the line passing through the two furthest points, we conclude the problem, since clearly all other points will be inside this area.

Solution to P6 :

Solution submitted by `@Bernat#9585` on discord :

We will find an algorithm that works in $O(1)$. Let $2r+1=p$ and $2s+1=q$ where r and s are primes. Choose a random number a such that $a^4 \not\equiv 1 \pmod{pq}$. We know there are at most 4^2 values of a satisfying this so it takes at most 17 tries to ensure a number satisfying the above condition. If $a \mid n$, we are done since $a=p$ or $a=q$. Thus suppose this is not the case and $\gcd(a,n)=1$. Now, we use the well-known fact that $\operatorname{ord}_n(a) \mid \varphi(n)=(p-1)(q-1)=4rs$ to get that $\operatorname{ord}_n(a^4) \in \{r,s,rs\}$.
For the first and second case, multiplying by 2 and adding one gives our desired p or q. For the second case, we write $4rs=(p-1)(q-1)=n-(p+q)+1$. Solving for $p+q$, we now know the sum and the product of the numbers p and q. Solving the quadratic $x^2-(p+q)x+pq=0$ will conclude the proof.
Remark: In order to distinguish between the first case and the second case, just check if $2\operatorname{ord}_n(a^4)+1 \mid n$. If it does not divide, then we are in the case rs. Otherwise, we have found either p, q or pq, but the third case is impossible since $2rs+1<(2r+1)(2s+1)$.
If $l \geq d$, then the $2l \times d$ rectangle can surround the entire backyard and we are done.
Now we consider the case of $l<d$. Suppose A and B represents two elves such that the distance between them are largest among any two elves. Also, let C be the elve such that among all the other elves, C has the maximal distance from AB. If A, B and C are collinear, we are done. If not, we can put the $2l \times d$ rectangle in a way such that AB is parallel to the side of the rectangle with a length of d and AB passes the middle line of the rectangle, we claim that the rectangle can surround all the elves. $\triangle ABC$ can be surrounded by a $l \times d$ rectangle. Since $\triangle ABC$ can be surrounded by a $l \times d$ rectangle, one of its altitudes has length at most l. Since AB is the longest side of the triangle, the corresponding altitude has to it must be shortest among the three altitudes, it's length is at most l. So the distance

between C and AB is at most l. But among all the other elves, C has the maximal distance from AB, so the distances between AB and any elves are at most l. All the elves can be surrounded by a $2l \times d$ rectangle.