IGMO 2021 Christmas Edition

25th December 2021

Question	Points
1	7
2	7
3	7
4	7
5	7
6	7
Total	42

Comments : 2 Pages, 3 Problems per page. Some questions may have a note at the bottom clarifying any ambiguity or confusion in the wording of the questions, they are not meant to be taken as hints.

ROUND 2 QUESTIONS

Problem 1 :

Santa Claus decorates his Christmas tree with a decoration which has a shape of a regular 12-sided polygon. Let the 12-sided polygon be $A_1A_2A_3...A_{12}$. Suppose I_1, I_2 and I_3 are the incentres of $\triangle A_1A_2A_5$, $\triangle A_5A_7A_8$ and $\triangle A_8A_{11}A_1$ respectively. Prove that I_1A_8, I_2A_1 and I_3A_5 are concurrent.

Problem 2 :

Santa has almost finished decorating his giant gingerbread house for Christmas. The only thing left to do is to create a circular fence around it. For this purpose Santa wants to use $n \geq 2$ candy canes in 3 colors: Green, Red and White, but he doesn't want any two adjacent candy canes to have the same color. Find the number of possible arrangements of this fence in terms of n

Note : Single candy canes are distinguishable.

Problem 3 :

For $n \geq 2$, let $a_1, a_2, \ldots, a_n$ be reals such that $a_1 + a_2 + \cdots + a_n = n^n - 1$.

Show that

$$a_1^2 + \frac{a_2^2}{1+a_1^2} + \cdots + \frac{a_n^2}{1+a_1^2+a_2^2+\cdots+a_{n-1}^2} > n\left(\frac{n^2}{\sqrt[n]{n+1}} - 1\right)$$

Turn over for remaining problems

Problem 4 :

Let f and g be real-valued functions defined for all real numbers x and a, and s, m be some positive constants, such that f, g satisfy the equations

1. $f(x+a) + f(x-a) = \dfrac{2f(x)g(a)}{s}$
2. $|f(x)| \leq m$

for all x, a. Prove that if $|f|$ is not identically zero, and attains a maximum value, then $|g(a)| \leq s$ for all a.

Problem 5 :

There are some (at least 3) elves in Santa's backyard. The backyard has a circular shape with diameter d. Santa finds that any three elves can be surrounded by an $\ell \times d$ rectangle. Prove that all the elves can be surrounded by a $2\ell \times d$ rectangle.

Note: An elf being "surrounded" by a rectangle means that the point corresponding to the elf is contained within the rectangle, or is on its perimeter. The elves have no areas, they are points.

Problem 6 :

For Christmas, Santa gifts us a special machine. This special machine takes as input any relatively prime positive integers a, n and returns the order of a modulo n as the output, that is to say : returns the least positive integer b such that $a^b \equiv 1 \mod n$. Using this special machine, devise an algorithm of time complexity at most $O(\log(n))$ to factorize natural numbers n of the form pq, where p, q are safe primes(which means p,q,$\frac{p-1}{2}$,$\frac{q-1}{2}$ are all primes greater than 5).

Note : You should assume that calling the special machine is $O(1)$, and for two positve integers a and b, calculating $\gcd(a, b)$ is $O(\log(\max(a, b))$.