# IGMO 2020 Round 1

*18th December 2020*

| Question | Points |
|---|---|
| 1 | 7 |
| 2 | 7 |
| 3 | 7 |
| 4 | 7 |
| 5 | 7 |
| 6 | 7 |
| Total | 42 |

**Instructions:**

1. This examination contains 3 pages, including this page.
2. You have **twelve (12) hours** to submit your solutions starting from when you accessed the paper.
3. Submit your answers in the form that came with the E-Mail. If your solutions are written, you are asked to scan your answers using CamScanner or TapScanner. If your solutions are typed using LaTeX(not recommended), then you are asked to send the solutions to **each** question as a separate PDF (or a screenshot of each answer) in the corresponding submission section
4. You are not allowed to disclose any questions on any online forums until 20:00 GMT 19th December. Do not participate or attempt the paper along with someone else, each contestant should be individual.

# ROUND 1 QUESTIONS

## Problem 1 :

For any natural number $n$ and all natural numbers $d$ dividing $2n^2$ show that $n^2 + d$ is not the square of a natural number

## Problem 2 :

A computer calculates the $n$th Fibonacci Number ($F_n$, where $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$) using "steps". A step is defined to be a single calculation (this means carrying out $a + b$ when we already know $a$ and $b$ counts as one step, though if we don't know what $a$ or $b$ are, carrying out $a + b$ takes *the number of steps to find* $a$ + *the number of steps to find* $b$ + 1, with the 1 coming from calculating $a + b$). The computer can only have $F_0$ and $F_1$ permanently stored, and it has to calculate everything else all over again when a request to calculate a new Fibonacci number is made. The advantage of this process is that the **final addition step** (to add one due to calculating $a + b$) gets omitted(by some algorithmic magic). In this algorithm, the steps needed to "call" $F_0$ and $F_1$ (these are the only values for which a call counts as a step) are both 1. Given that the computer can only carry out simple addition of 2 numbers at most :

- Find an expression for $S(F_n)$, the number of steps taken to find $F_n$ using this approach if we can only have $F_0$ and $F_1$ stored for all $n \geq 2$ where $S(F_0) = 1$ and $S(F_1) = 1$ due to the "calling" feature
- Find an expression for $\gcd(S(F_n), S(F_m)))$ involving only $n, m$ and the gcd function for $n, m \geq 2$

Now, the computer has the ability to "cache" (store) all previously calculated Fibonacci Numbers, but the final step is not omitted anymore. Assuming that we've already calculated $F_k$ for some $2 \leq k \leq n - 1$, and the probability of picking an $F_k$ is equally likely for all $k$ :

- Show that the expected number of steps taken to calculate $F_n$, $\mathbb{E}(S(F_n))$, using the "cache" feature (if we have $F_0$ and $F_1$ stored and there is no calling step) is $\frac{n-1}{2}$

Note : $\mathbb{E}(X) = \sum P(X = x_i)x_i$ here where $x_i$ is any possible value $X$ can take. Also note that the computer does not know that $F_2 = F_1$ or that $F_0 = 0$

## Problem 3 :

Given that $x_1, x_2 \cdots, x_k$ are positive reals such that $\displaystyle\sum_{i=1}^{k} x_i^{n-1} = k - 1$, prove that

$$\frac{x_1^n}{x_2 + x_3 + \cdots + x_k} + \frac{x_2^n}{x_1 + x_3 + \cdots + x_k} + \cdots + \frac{x_k^n}{x_1 + x_2 + \cdots + x_{k-1}} \geq 1$$

## Problem 4 :

Three frogs are initially on the vertices of an equilateral triangle with sides length of 1. The frogs can jump over each other in the following way: if frog $A$ at point $M$ jumps over frog $B$ at point $N$, then frog $A$ will land on point $O$ such that $MN = ON$ and $M$, $N$, $O$ are co-linear. By repeated jumping, is it possible that the three frogs eventually move to the vertices of an equilateral triangle with sides length of 10?

## Problem 5 :

Let $I$, $H$, $O$ be the incentre, orthocentre and circumcentre of $\triangle ABC$ respectively. $D$ is the circumcentre of $\triangle AIC$. $H$ is reflected along $BC$ and $AB$ to $E$ and $F$ respectively. Prove that $D$, $O$, $F$ are collinear if and only if $DE$ is perpendicular to $EF$.

## Problem 6 :

Some points are drawn on a plane such that the points do not have equal distances to each other. For each point, a line is drawn to connect it with its nearest point. Find the maximum possible number of lines that a point is connected with.