Global and Multiculutral Mathematical Association

International Gamma Mathematical Olympiad 2022

Round 1

21st May 2022 TIME ALLOWED: 4 h 15 m

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIFTEEN (15)** integer answer questions and **ONE (1)** proof based question and a total of **FIVE (5)** pages. Positive integers refers to the set $\{1, 2, 3, \ldots, \}$
2. For questions 1-15. Enter an integer between 0-999 inclusive (so there are 1000 acceptable options). There is no negative marking.
3. For the proof based problem, submit a properly justified proof. If you are submitting a handwritten proof (recommended), please scan it using CamScanner or TapScanner or any scanning app and submit the scanned version.
4. You are ONLY allowed a 4 function calculator. The questions are designed such that computer aid will not help with the problems.
5. Please submit all your answers within 4 hrs 15 mins of you opening the paper. Do not share your answers with anyone else. Do not ask for help on online forums, we are monitoring them.

Best of Luck!

Question 1. (2 marks)

a_1, a_2, ..., a_8 are 8 distinct positive integers. b_1, b_2, ..., b_8 are another 8 distinct positive integers (a_i, b_j are not necessarily distinct for $i, j = 1, 2, ...8$). Enter the smallest possible value of $a_1^2b_1 + a_2^2b_2 + ... + a_8^2b_8$.

Question 2. (2 marks)

A frog can jump over a point in the following way: if a frog at point M jumps over point N, it lands on the point of reflection of point M over N. A frog is initially at point X. A, B, C are points such that X, A, B and C are on the same plane. The frog first jumps over A, then over B, then over C, and then continues to jump over point A, B and C alternately. Suppose $XA = 36$cm, $XB = 84$cm, $XC = 48$cm. After 2021 jumps, the frog is at point Y. $XY = k$ cm. Enter k.

Question 3. (2 marks)

Enter the smallest positive integer n such that any set of n relatively prime integers strictly greater than 1 and strictly less than 2022 contains at least 1 prime number.

Question 4. (2 marks)

We define $a(n)$ as the number of positive divisors of n and $b(n)$ as the sum of the positive divisors of n. There are m positive integers n between 1 and 2022 inclusive such that both $a(n)$ and $b(n)$ are odd. Enter m

Question 5. (2 marks)

We call a coloring of a 2022×2022 checkerboard a **banger** coloring if it is colored by 2 colors, black and white, such that each 2×2 square contains an even number of black cells. The number of banger colorings is b. Let $b = b_1b_2 \cdots cd$ (so d is the last digit of b and c is the 2nd last digit of b). Enter $c^2 + d^2 + 5$

Question 6. (2 marks)

Let w be an arbitrary complex number with a non-zero imaginary part and a non-zero real part. There are n values of $k \in \{0, 1, 2, 3, 4, \ldots, 999\}$ for which

$$z = \left(\bar{w}w^3 - w\bar{w}^3\right)^k$$

is a real number, Enter n.

Question 7. (2 marks)

Let $s(n)$ be defined as the sum of digits of n. Enter the sum of all positive integers n for which

$$s(11^n) = 2^n$$

Question 8. (2 marks)

A subset of $\{1, 2, ..., 12\}$ is called trivoidant if no two elements of the subset differ by 3. Enter the number of trivoidant subsets of $\{1, 2, ..., 12\}$

Question 9. (4 marks)

Let n be a positive integer and $s(n)$ be defined as the sum of digits of n. Also, let

$$q(n) := \frac{s(2n)}{s(n)}$$

If a is the smallest possible value of $q(n)$ and b is the biggest possible value of $q(n)$, then enter $100(a + b)$

Question 10. (4 marks)

There are n points in a 3-dimensional space. If two points are exactly 1 unit from each other, a line is joint between them. Let the number of total lines being joint be l. Suppose the maximum possible value of $\frac{l}{n^2}$ is $\frac{p}{q}$, where p and q are positive integers that are relatively prime to each other. Enter $100p + q$.

Question 11. (4 marks)

Let $f(n)$ denote the largest positive integer m such that 7^m divides $H(n)$ where $H(n) = 1^1 \cdot 2^2 \cdots n^n$. As n gets arbitrarily large, $f(n) \sim kn^2$. Enter $a^2 + b^2$ where $k = \frac{a}{b}$, a, b are positive integers and $\gcd(a, b) = 1$. For those unfamiliar of $\sim$, equivalently

$$k = \lim_{n \to \infty} \frac{f(n)}{n^2}$$

Question 12. (4 marks)

Let ω be the incircle of $\triangle ABC$. Suppose ω touches BC, CA, AB at points D, E, F respectively. AD, BE, CF intersect with ω at points G, H, I respectively. HI and AD intersect at point X. If $GH = 8$, $HI = 9$, $IG = 10$, $IX = \frac{p}{q}$, where p, q are positive integers that are relatively prime to each other. Enter $p + q$.

Question 13. (4 marks)

30 people stand on a circle with radius of 1 unit. In view of the COVID-19 outbreak, the government has implemented social distancing policy. The distance between 2 people should be more than 1 unit, otherwise they will be fined. The amount of fine for each person is equal to \$(number of people that are within 1 unit from him/her). For example, if 3 people have distances of less than or equal to 1 unit from person A, then person A has to pay a fine of \$3. Enter the minimum total fine (in dollars) that has to be paid by all the 30 people. (Assume that the peoples are points, ie they have no areas)

Note: You should only enter the amount, do not include the dollar sign. For example, if the minimum is \$14, enter 14.

Question 14. (4 marks)

For real numbers $\theta_1, \theta_2, \ldots \theta_{11}$ satisfying

$$\sum_{i=1}^{11} \sin(\theta_i) = 1$$

The minimum value of

$$\sum_{i=1}^{11} \sin(3\theta_i)$$

is $-\frac{a}{b}$ where a, b are positive integers and $\gcd(a, b) = 1$. Enter $a - b$

Question 15. (4 marks)

Let D be the centre of circle which passes through the mid-points of the three edges of $\triangle ABC$, E and F be points on AB and AC respectively such that $DE \perp AB$ and $DF \perp AC$. It is known that $AB = 15$, $AC = 20$, $4DE = 3DF$. $BC^2 = q + r\sqrt{s}$, where q and r are integers and s is a square free integer. Enter $q + r + s$.

<u>Proof Based Question</u> (8 marks)

Find all pairs (n, m) of positive integers such that

$$9^n - 5 \cdot 2^m = 1$$

Show that you have found all such pairs.

Note: Simply listing all pairs is not enough, you will have to submit a proof that there are no pairs other than the ones you have mentioned. Partial credit will be given.

END OF PAPER