IGMO 2022 Round 2 Day 2

3rd July 2022

Question	**Points**
4	7
5	7
6	7
Total	21

Instructions:

1. This examination contains 2 pages, including this page.
2. You have **4 hours 30 minutes** to submit your solutions starting from when you accessed the paper. The question paper access and submission links are unique. Do **NOT** share them.
3. Submit your answers in the form that came with the E-Mail. If your solutions are written, you are asked to scan your answers using CamScanner or TapScanner. If your solutions are typed using LaTeX(not recommended), then you are asked to send the solutions to **each** question as a separate PDF (or a screenshot of each answer) in the corresponding submission section.
4. You are ONLY allowed a 4 function calculator. Simply stating answers will not give you points, you have to provide full proofs for each problem. You can submit multiple times, and only the latest submission (within the 4.5 hour time period) will be marked.
5. There is a live clarifications Google Doc that will be updated with clarifications regarding the wording of questions. If you are confused by the wording of a question, check this document. If you do not find the clarification here, message Creative Math on discord.
6. You are not allowed to disclose any questions on any online forums until 04:30 GMT 4th July. Do not participate or attempt the paper along with someone else, each contestant should be individual.

DAY 2 QUESTIONS

Problem 4 :

Let's take any positive integer $n_0 \in \{1, 2, 3 \ldots\}$. Then, let n_1 be equal to the sum of digits of the square of n_0. Repeat this process to obtain n_2 from n_1, n_3 from n_2 and so on. In this way we obtain a sequence of natural numbers denoted as $(n_0; n_1, n_2, ...)$. For example when we pick $n_0 = 7$ we obtain

$$7 \rightarrow 4 + 9 = 13 \rightarrow 1 + 6 + 9 = 16 \rightarrow ...$$

which creates sequence

$$(7; 13, 16, ...)$$

If in the $(n_0; n_1, n_2, ...)$ sequence we are able to find two indices i and j such that $i > j$ and $n_i = n_j$, then we say that this sequence "***gets caught in a loop***". If in the $(n_0; n_1, n_2, ...)$ sequence we are able to find an index $i > 0$ such that $n_i = n_0$, then we call this sequence "***a loop***". Find all possible loops and prove that for every n_0, the sequence $(n_0; n_1, n_2, ...)$ gets caught in a loop.

Problem 5 :

Let O be a fixed point on a plane. P_1, P_2, ... , P_{2022} are 2022 variable points on the same plane which do not coincide with O. Initially, P_1 is an arbitrary point and the line segment OP_{n+1} is $\frac{\pi}{1011}$ anticlockwise to the line segment OP_n for $n = 1, 2, ..., 2021$. In other words, $\angle P_n O P_{n+1} = \frac{\pi}{1011}$ in directed angles. A group of points is said to be "perfect" if O is the point such that the sum of the distances from that point to all the points in the group is the smallest possible. For example, the group of points formed by P_1, P_2, P_3, P_4, P_5 is said to be "perfect" if $OP_1 + OP_2 + OP_3 + OP_4 + OP_5 \leq XP_1 + XP_2 + XP_3 + XP_4 + XP_5$ for any point X on the plane.

(a) *(2/7 points)* Prove that the group of points formed by P_1, P_2, ..., P_{2022} is "perfect" initially.

(b) *(5/7 points)* You are given a task to remove the points P_1, P_2, ..., P_{2022} according the following rules. In each step, you could remove one point, and then rotate another point anticlockwise for $\frac{\pi}{3}$ around point O. After each step, the group formed by the remaining points on the plane must remain to be "perfect". Devise an algorithm to remove 1348 points from the initial 2022 points and prove that it works.

Problem 6 :

Let $n \in \{1, 2, \ldots\}$ and x, y, z all greater than 0 be real numbers such that $x + y + z = 3$. Prove that

$$(\sqrt{x} + 1)^n + (\sqrt{y} + 1)^n + (\sqrt{z} + 1)^n - n(xy + yz + zx) \geq 3(2^n - n)$$