IGMO 2022 Christmas Edition

25th December 2022

Question	Points
1	7
2	7
3	7
4	7
5	7
6	7
Total	42

Comments : 2 Pages, 3 Problems per page. Some questions may have a note at the bottom clarifying any ambiguity or confusion in the wording of the questions, they are not meant to be taken as hints.

ROUND 2 QUESTIONS

Problem 1 :

Prove that for all positive real numbers x, m, a, s,

$$6 + 6^{x+m} + 6^{x+m+a} + 6^{x+m+a+s} > \frac{1}{2}(6^{x+1}) + \frac{1}{2}(6^{m+1}) + \frac{1}{3}(6^{a+1}) + 6^{s}$$

Problem 2 :

Santa and an invisible elf play a hide and seek game in the Euclidean plane. Firstly, the elf chooses 3 points, A_1, A_2 and A_3. These points are known to Santa. Also, we define $A_s = A_{s-3}$ for all $s \geq 4$. Then the elf chooses a point P_0 such that the distant between P_0 and A_1 is 100. P_0 is his original position, and it is not known to Santa.

In the beginning of round n, Santa chooses a number θ_n between 45 to 90, and then the elf will move to point P_n, which is defined as the point where P_{n-1} is rotated θ_n^o anti-clockwise about A_n. The point is not known to Santa since the elf is invisible. Santa will then choose an area to scan using an elf detector. The detector can scan a circular area of radius of 1. If the invisible elf is within the area of scanning (inside the circle or on the edge of the circle) of the detector, then Santa wins.

Does there exist a winning strategy to ensure that Santa can win within 2022 rounds? Prove your claim.

Note : Assume the invisible elf is a point, i.e. he has no area.

Problem 3 :

Santa decorates his Christmas tree with a triangular decoration. Suppose the triangular decoration can be represented by $\triangle ABC$. Let ω be its incircle and ω_A, ω_B, ω_C be its A-, B-, C-excircles respectively. Let J_A, J_B, J_C be the A-, B-, C-excentres of $\triangle ABC$ respectively. X is the radical centre of ω, ω_B, ω_C. Y is the radical centre of ω, ω_C, ω_A. Z is the radical centre of ω, ω_A, ω_B. Prove that XJ_A, YJ_B, ZJ_C are concurrent.

Turn over for remaining problems

Problem 4 :

Because of inflation, Santa can't afford buying gifts for all children this year. He decided to divide his ordered list of good children into those who will get a gift and those who won't. To be as fair as possible, he came up with the following (seemingly random) rule:

"Each child has its own number on my list. If n is a positive integer which satisfies

$$\tau(n^k) \leq k \cdot \tau(n)$$

for all positive integers $k \geq 2$, then the n^{th} child from my list will get a gift."

Characterize all positive integers n such that the n^{th} child from the list gets a gift this year.

Note: $\tau(n)$ denotes the number of positive divisors of n.

Problem 5 :

Santa draws a Christmas tree in the following way. He first draws an acute-angled triangle $\triangle ABC$. He then lets M be the mid-point of BC, A' be the point of reflection of A over BC, D be the point of intersection of line segment AM and the circumcircle of $\triangle A'BC$, E and F be points on AB and AC respectively such that D, E, F are collinear. Prove that $\frac{AE}{ED \cdot DB} = \frac{AF}{FD \cdot DC}$.

Note: Line segments AE, ED, DB, BC, CD, DF, FA form the shape of a Christmas tree!

Problem 6 :

After delivering all the Christmas presents, Santa finally have some leisure time to do Maths, which is his favourite hobby. Santa proposes two new sequences: Christmas sequence and Santa sequence. Numbers in the Christmas sequence are known as Christmas numbers.

The Christmas sequence is defined as:

$C_0 = 0$, $C_1 = 1$, $C_{n+1} = 2022C_n + C_{n-1}$ for $n \geq 1$.

The Santa sequence is defined as:

$S_0 = 2$, $S_1 = 2022$, $S_{n+1} = 2022S_n + S_{n-1}$ for $n \geq 1$.

Santa finds 4043 children and labels them from 1 to 4043. He asks the n^{th} child to express $C_1S_{2023} + C_2S_{2024} + ... + C_{2022}S_{4044}$ as a sum of n non-zero Christmas numbers. Those who can do so can get an extra gift, which is a cute Christmas frog. Amongst the 4043 children, who can potentially get an extra gift? Prove your claim.