IGMO Christmas Edition Solutions

Note:

This was an unofficial version released by the team. Hence there are no submission statistics. However, we did encourage users from the discord server affiliated with IGMO (IΓMC) to send their solutions, and wherever an appropriate solution was received, the user who submitted it was mentioned. If the official solution was vastly different from the submitted solution, it was also included. The affiliated discord server is linked on the website

Problem 1 :

Prove that for all positive real numbers x, m, a, s,

$$6+6^{x+m}+6^{x+m+a}+6^{x+m+a+s}>\frac{1}{2}(6^{x+1})+\frac{1}{2}(6^{m+1})+\frac{1}{3}(6^{a+1})+6^{s}$$

Solution

Problem 2 :

Santa and an invisible elf play a hide and seek game in the Euclidean plane. Firstly, the elf chooses 3 points, A_1, A_2 and A_3. These points are known to Santa. Also, we define $A_s=A_{s-3}$ for all $s\geq 4$. Then the elf chooses a point P_0 such that the distant between P_0 and A_1 is 100. P_0 is his original position, and it is not known to Santa.

In the beginning of round n, Santa chooses a number θ_n between 45 to 90, and then the elf will move to point P_n, which is defined as the point where P_{n-1} is rotated θ_n^o anti-clockwise about A_n. The point is not known to Santa since the elf is invisible. Santa will then choose an area to scan using an elf detector. The detector can scan a circular area of radius of 1. If the invisible elf is within the area of scanning (inside the circle or on the edge of the circle) of the detector, then Santa wins.

Does there exist a winning strategy to ensure that Santa can win within 2022 rounds?
Prove your claim.

Note : Assume the invisible elf is a point, i.e. he has no area.

Solution

Problem 3 :

Santa decorates his Christmas tree with a triangular decoration. Suppose the triangular decoration can be represented by $\triangle ABC$. Let ω be its incircle and ω_A, ω_B, ω_C be its A-, B-, C-excircles respectively. Let J_A, J_B, J_C be the A-, B-, C-excentres of $\triangle ABC$ respectively. X is the radical centre of ω, ω_B, ω_C. Y is the radical centre of ω, ω_C, ω_A. Z is the radical centre of ω, ω_A, ω_B. Prove that XJ_A, YJ_B, ZJ_C are concurrent.

Solution

Problem 4 :

Because of inflation, Santa can't afford buying gifts for all children this year. He decided to divide his ordered list of good children into those who will get a gift and those who won't. To be as fair as possible, he came up with the following (seemingly random) rule:
"Each child has its own number on my list. If n is a positive integer which satisfies

$$\tau(n^k) \leq k \cdot \tau(n)$$

for all positive integers $k \geq 2$, then the n^{th} child from my list will get a gift."

Characterize all positive integers n such that the n^{th} child from the list gets a gift.

Note: $\tau(n)$ denotes the number of positive divisors of n.

Solution

Problem 5 :

Santa draws a Christmas tree in the following way. He first draws an acute-angled triangle $\triangle ABC$. He then lets M be the mid-point of BC, A' be the point of reflection of A over BC, D be the point of intersection of line segment AM and the circumcircle of $\triangle A'BC$, E and F be points on AB and AC respectively such that D, E, F are collinear. Prove that

$$\frac{AE}{ED \cdot DB} = \frac{AF}{FD \cdot DC}$$

Note: Line segments AE, ED, DB, BC, CD, DF, FA form the shape of a Christmas tree!

Solution

Problem 6 :

After delivering all the Christmas presents, Santa finally have some leisure time to do Maths, which is his favourite hobby. Santa proposes two new sequences: Christmas sequence and Santa sequence. Numbers in the Christmas sequence are known as Christmas numbers.
The Christmas sequence is defined as:

$$C_0 = 0, C_1 = 1, C_{n+1} = 2022C_n + C_{n-1} \text{ for } n \geq 1$$

The Santa sequence is defined as:

$$S_0 = 2, S_1 = 2022, S_{n+1} = 2022S_n + S_{n-1} \text{ for } n \geq 1$$

Santa finds 4043 children and labels them from 1 to 4043. He asks the n^{th} child to express $C_1S_{2023} + C_2S_{2024} + ... + C_{2022}S_{4044}$ as a sum of n non-zero Christmas numbers. Those who can do so can get an extra gift, which is a cute Christmas frog. Amongst the 4043 children, who can potentially get an extra gift? Prove your claim.

Solution

Solution to P1 :

Problem posed by @pepemaths

Solvers: @bagbagnagee from Instagram, Culver Kwan (@culverkwan)

Official Solution:
Since x, m, a, s are positive, $6^x > 1$, $6^m > 1$, $6^a > 1$, $6^s > 1$. Let $6^x = 1 + \alpha$, $6^m = 1 + \beta$, $6^a = 1 + \gamma$, $6^s = 1 + \delta$, where α, β, γ, δ are positive.

$6 + 6^{x+m} + 6^{x+m+a} + 6^{x+m+a+s} - \frac{1}{2}(6^{x+1}) + \frac{1}{2}(6^{m+1}) + \frac{1}{3}(6^{a+1}) + 6^s$
$= 6 + (1+\alpha)(1+\beta) + (1+\alpha)(1+\beta)(1+\gamma) + (1+\alpha)(1+\beta)(1+\gamma)(1+\delta) - 3(1+\alpha) - 3(1+\beta) - 2(1+\gamma) - (1+\delta)$
$= \alpha\beta\gamma\delta + 2\alpha\beta\gamma + \alpha\beta\delta + 3\alpha\beta + \alpha\gamma\delta + 2\alpha\gamma + \alpha\delta + \beta\gamma\delta + 2\beta\gamma + \beta\delta + \gamma\delta > 0$

So $6 + 6^{x+m} + 6^{x+m+a} + 6^{x+m+a+s} > \frac{1}{2}(6^{x+1}) + \frac{1}{2}(6^{m+1}) + \frac{1}{3}(6^{a+1}) + 6^s$.

Solution to P2 :

Solved by @Dirichlet #6744 from Discord and Culver Kwan (@culverkwan)

Solution for P2 (By @Dirichlet #6744 from Discord)

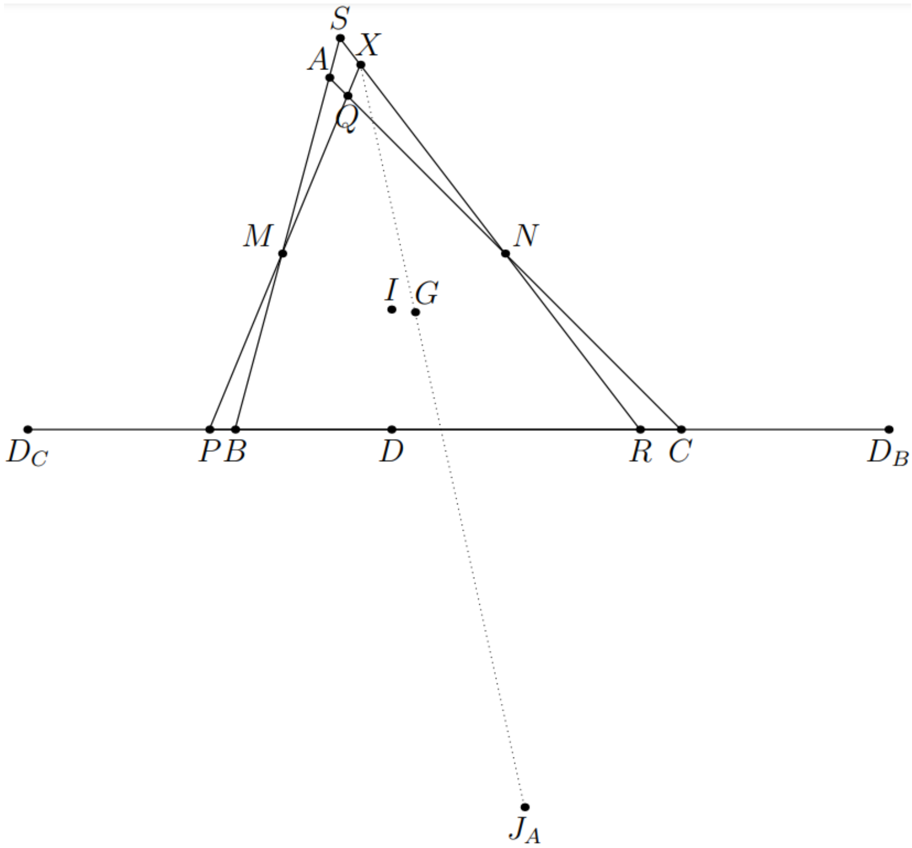
Let C_n be the image of C_{n-1} under rotation by θ_n^o about A_n, with C_0 being the circle of radius 100 centered at A_1. Then let d_n be the subset of C_n which has been ruled out by the detector. Note that if $x \in d_{n-1}$, then $R_{\theta_n, A_n}[x] \in d_n$. Finally, let $S_n = C_n - d_n$. Then S_n is the set of possible positions of P_n, given the θ_i and A_i. Since C_n is determined by the θ_i and A_i, and the d_i are Santa's choice, Santa can completely determine S_n for all n. By using the detector to check consecutive circular arcs, S_n is a circular arc with length $L(S_n) \leq L(S_{n-1}) - 2 \leq L(S_0) - 2n = 200 - 2\pi$. For all $n > 100\pi$, we have that $200 - 2\pi < 0$, so $L(S_n) < 0$. Obviously this is impossible, so it must be that $L(S_{\lfloor 100\pi \rfloor}) < 2$, $S_{\lceil 100\pi \rceil} = \phi$. Thus after 315 turns, there are no undetected positions for P_{315}, so the game has ended and Santa has won.

Solution to P3 :

Problem posed by Hakan Karakus

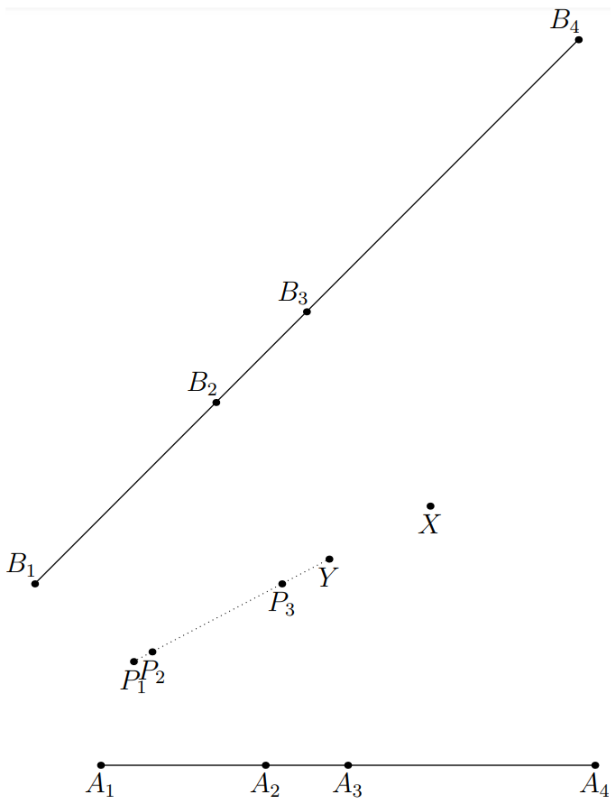
Solvers: @ChristopherPi #8528 from Discord , YakamozKan

Official Solution by Proposer, Hakkan Karakush
Let M and N be midpoints of AB and AC respectively. Let ω touch sides BC, AC, AB at D, E, F respectively. Let ω_C touch sides BC, AC, AB at D_C, E_C, F_C. Let ω_B touch sides BC, AC, AB at D_B, E_B, F_B. X lies on the radical axis of ω and ω_B. Radical axis of ω and ω_B is the line passing through midpoints of DD_C, EE_C, FF_C. Similarly X also lies on the line passing through midpoints of DD_B, EE_B, FF_B. Midpoint of FF_C is M, and midpoint of EE_B is N. Let midpoints of DD_C, EE_C, DD_B, FF_B be P, Q, R, S respectively. X is the intersection of lines PMQ and RNS.



Lemma. Let A_1, A_2, A_3, A_4 be collinear points. Let B_1, B_2, B_3, B_4 be collinear points. Then, points $A_iB_j \cap A_jB_i$ all lie on the same line if $(A_1, A_2; A_3, A_4) = (B_1, B_2; B_3, B_4)$

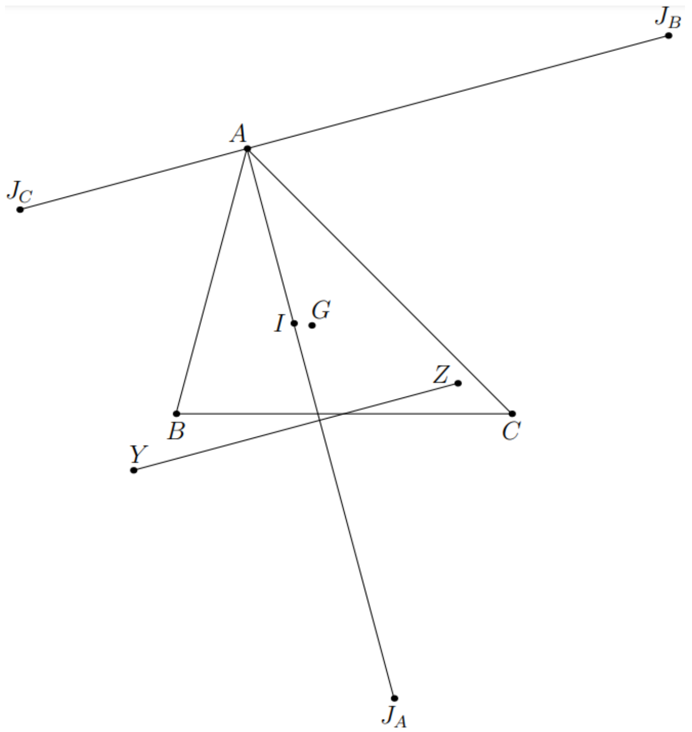
Proof. Let $A_1B_2 \cap A_2B_1 = P_1$, $A_1B_3 \cap A_3B_1 = P_2$, $A_2B_3 \cap A_3B_2 = P_3$. By Pappus theorem, P_1, P_2, P_3 are collinear, call this line l. Let $A_3B_4 \cap l = X_1$, $A_4B_3 \cap l = X_2$. Let $A_3B_3 \cap l = Y$. From B_3 to l, we get $(A_1, A_2; A_3, A_4) = (P_2, P_3, Y, X_1)$. From A_3 to l, we get $(B_1, B_2; B_3, B_4) = (P_2, P_3, Y, X_2)$. Therefore, $X_1 = X_2 = X = A_3B_4 \cap A_4B_3 \in l$.



Let $AB \cap CJ_A = U$, $AC \cap BJ_A = V$. We will use the lemma for points U, B, M, S and points V, C, N, Q, which proves $J_A = UC \cap VB$, $G = MC \cap NB$, $X = MQ \cap NS$ are collinear. We just need to show that $\frac{UB}{MB}\frac{MS}{US} = \frac{VC}{NC}\frac{NQ}{VQ}$. We know that $UB = \frac{ac}{b-a}$, $MB = \frac{c}{2}$, $MS = \frac{a}{2}$, $US = \frac{ac}{b-a} + \frac{a}{2} + \frac{c}{2}$, $VC = \frac{ab}{c-a}$, $NC = \frac{b}{2}$, $NQ = \frac{a}{2}$, $VQ = \frac{ab}{c-a} + \frac{a}{2} + \frac{b}{2}$. Putting these equalities together, we get the desired result that proves $(U, M; B, S) = (V, N; C, Q)$. XJ_A, YJ_B, ZJ_C pass through G.

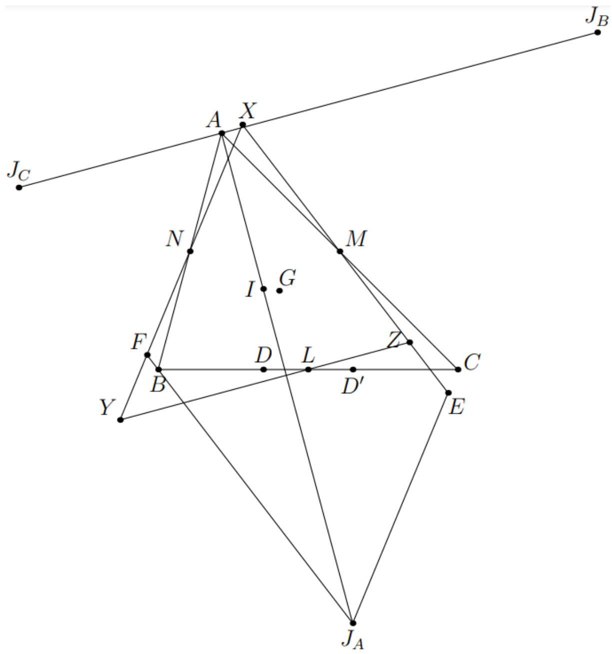
Solution by @ChristopherPi #8528, YakamozKan

YZ is the radical axis of w and w_A, which is perpendicular to the line joining their centers, namely IJ_A (I the incenter of ABC), while J_A lies on the internal bisector of $\angle BAC$ and J_B, J_C on the external bisector, so YZ perpendicular to IJ_A perpendicular to J_BJ_C. Hence XYZ, $J_AJ_BJ_C$ are perspective at the line at infinity, so by Desargues are perspective at a point (hence XJ_A, YJ_B, ZJ_C are concurrent).



Intersection is G by @ChristopherPi #8528

We show that X, G, J_A are collinear (G is the centroid of ABC). Let the midpoints of BC, CA, AB be L, M, N; let the incircle and A-excircle touch BC at D, D'. It's known that $DL = D'L$, and both of these are tangents to respective circles, so L has equal powers with respect to w and w_A, and so the radical axis of w and w_A is the line through L perpendicular to AI; hence we define X to be the intersection of ℓ_1 the line through M perpendicular to BI and ℓ_2 the line through N perpendicular to CI. Also let ℓ_1 meet CJ_A at E and let ℓ_2 meet BJ_A at F. Since $BF \perp BI \perp ME$, $FN \perp CI \perp CE$, and NM is parallel to CB, by the converse of Pascal on $BFNMEC$ ($BF \cap ME$, $FN \cap EC$, $NM \cap CB$ are all points at infinity, collinear on the line at infinity) we have that B, C, E, F, M, N lie on a common conic; then by Pascal on $BMECNF$, $BM \cap CN$, $ME \cap NF$, $EC \cap FB$ are collinear, but these are precisely G, X, J_A, so X, G, J_A are collinear and by symmetry G lies on YJ_B, ZJ_C.



Solution to P4 :

Problem posed by @mathinity

Solvers: @bagbagbagee, @aidancheung005, @culverkwan, @guisesurico, @esmir.z1

Official Solution: Firstly lets consider $n = p^a$, where p is a prime number and $a \in \mathbb{N}$. Then for $k \geq 2$ we have

$$\tau(n^k) = ka + 1 \leq ka + k = k(a+1) = k\tau(n),$$

so n^{th} child gets a gift. Now we can assume that n has at least two distinct prime factors. Therefore

$$n = p^a q^b m, \text{ where } m \text{ is coprime with } p \text{ and } q.$$

Then we have

$$\tau(n^k) = (ka+1)(kb+1)\tau(m^k)$$

and

$$k \cdot \tau(n) = k(a+1)(b+1)\tau(m).$$

We clearly have $\tau(m^k) \geq \tau(m)$, so n satisfies $\tau(n^k) \leq k \cdot \tau(n)$ if we have

$$(ka+1)(kb+1) \leq k(a+1)(b+1),$$

but that's not true, since we can transform it into
$k(k-1)ab \leq k-1$, which is definitely not true, because $k \geq 2$. Finally only 1 and p^a numbers will receive gifts.

Solution by @giusesurico (Giuseppe Surico)
To start things off, we will first investigate what happens with the powers of a prime p. After that, we will find a general solution.
Obviously, $\tau\left((p^n)^k\right) = nk+1$ and $k\tau(p^n) = k(n+1)$, so it is pretty obvious that $nk+1 \leq k(n+1)$ for every $k \geq 1$.
More generally speaking, using the fundamental theorem of arithmetics, we get that $n = p_1^{h_1} \dots p_t^{h_t}$.
This, in turn doing some combinatorics, implies that

$$\tau(n) = (1+h_1)(1+h_2)\dots(1+h_t)$$

In the same way, it is also true that $\tau\left(n^k\right) = (1+kh_1)(1+kh_2)\dots(1+kh_t)$. We now compute the limit

$$\lim_k \frac{\tau\left(n^k\right)}{k\tau(n)} = \frac{(1+kh_1)(1+kh_2)\dots(1+kh_t)}{k(1+h_1)(1+h_2)\dots(1+h_t)}$$

Clearly, if $t>1$, the limit goes to $+\infty$, telling us the inequality clearly won't hold for every k. In conclusion, the only integers satisfying the inequality for every k are the powers of every prime number.

Solution to P5 :

Problem posed by @pepemaths

Solved by @mathbreak from Instagram, Culver Kwan (@culverkwan)

Official Solution:

Let H be the orthocentre of $\triangle ABC$, then $\angle BHC = 180^o - \angle BAC$ (a well-known property of orthocentre). $\angle BHC + \angle BA'C = 180^o - \angle BAC + \angle BAC = 180^o$. Therefore B, A', C, H are concyclic. D is the intersection of the A-median and the circumcircle of $\triangle BHC$. So D is the point of projection of H on the A-median, also known as the "A-Humpty point" in the Olympiad community.

By property of Humpty point, $\angle BAM = \angle DBM$ and $\angle CAM = \angle DCM$. Let $\angle BAM = \angle DBM = x$, $\angle CAM = \angle DCM = y$.

Consider the areas of $\triangle AED$ and $\triangle AFD$, $\frac{[\triangle AED]}{[\triangle AFD]} = \frac{\frac{1}{2}(AE)(AD)\sin x}{\frac{1}{2}(AF)(AD)\sin y} = \frac{AE\sin x}{AF\sin y}$. Also, $\frac{[\triangle AED]}{[\triangle AFD]} = \frac{ED}{FD}$. So $\frac{AE\sin x}{AF\sin y} = \frac{ED}{FD}$.

Consider $\triangle DBC$, by sine rule, $\frac{DC}{DB} = \frac{\sin x}{\sin y}$.

Combining the two equations, we have $\frac{AE\cdot DC}{AF\cdot DB} = \frac{ED}{FD}$. Rearranging gives $\frac{AE}{ED\cdot DB} = \frac{AF}{FD\cdot DC}$.

Solution to P6 :

Problem posed by @pepemaths

Solved by Culver Kwan (@culverkwan)

Solution By Culver Kwan:

Only the 2022^{th} can potentially get the Christmas frog.

We first compute that $C_2 = 2022$ and $C_3 = 2022^2 + 1$.

Claim 1: For any $n \in \mathbb{N}$, $\sum_{i=1}^{n} C_{2i} = \frac{1}{2022}(C_{2n+1} - 1)$.

When $n=1$, $\sum_{i=1}^{1} C_{2i} = C_2 = 2022 = \frac{1}{2022}(2022^2+1-1) = \frac{1}{2022}\left(C_{2(1)+1} - 1\right)$. So the base case is true.

Assume Claim 1 is true for $n=k$. When $n=k+1$,

$$\begin{aligned}\sum_{i=1}^{k+1} C_{2i} &= \sum_{i=1}^{k} C_{2i} + C_{2k+2} \\ &= \frac{1}{2022}(C_{2k+1} - 1) + C_{2k+2} \\ &= \frac{1}{2022}(C_{2k+1} + 2022C_{2k+2} - 1) \\ &= \frac{1}{2022}\left(C_{2(k+1)+1} - 1\right)\end{aligned}$$

Hence via induction we have proved the claim. $\square$

Claim 2: For any $n \in \mathbb{N}$, $\sum_{i=1}^{2n} C_i S_{i+2022} = \frac{1}{2022} C_{2n} S_{2n+2023}$.

When $n = 1$,

$$\sum_{i=1}^{2(1)} C_i S_{i+2022} = C_1 S_{2023} + C_2 S_{2024} = S_{2023} + 2022 S_{2024} = S_{2025} = \frac{1}{2022} C_{2(1)} S_{2(1)+2023}$$

So the base case is true.

Assume Claim 2 is true for $n = k$. When $n = k + 1$,

$$\begin{aligned}
\sum_{i=1}^{2(k+1)} C_i S_{i+2022} &= \sum_{i=1}^{k} C_i S_{i+2022} + C_{2k+1} S_{2k+2023} + C_{2k+2} S_{2k+2024} \\
&= \frac{1}{2022} C_{2k} S_{2k+2022} + C_{2k+1} S_{2k+2023} + C_{2k+2} S_{2k+2024} \\
&= \frac{1}{2022} \left(C_{2k} S_{2k+2022} + 2022 C_{2k+1} S_{2k+2023} + 2022 C_{2k+2} S_{2k+2024} \right) \\
&= \frac{1}{2022} \left(C_{2k+2} S_{2k+2023} + 2022 C_{2k+2} S_{2k+2024} \right) \\
&= \frac{1}{2022} C_{2(k+1)} S_{2(k+1)+2023}
\end{aligned}$$

Hence via induction we have proved the claim. $\square$

The general terms of C_n and S_n are $\frac{1}{2\sqrt{1011^2+1}} \left(\alpha^n - \beta^n \right)$ and $\alpha^n + \beta^n$ where $\alpha = 1011 + \sqrt{1011^2 + 1}$ and $\beta = 1011 - \sqrt{1011^2 + 1}$, by some simple calculations. Notice that $\alpha\beta = -1$. By Claim 1 and Claim 2,

$$\begin{aligned}
\sum_{i=1}^{2022} C_i S_{i+2022} &= \frac{1}{2022} C_{2022} S_{4045} \\
&= \frac{1}{2022} \left(\frac{1}{2\sqrt{1011^2+1}} \left(\alpha^{2022} - \beta^{2022} \right) \right) \left(\frac{1}{2\sqrt{1011^2+1}} \left(\alpha^{4045} - \beta^{4045} \right) \right) \\
&= \frac{1}{2022} \left(\frac{1}{2\sqrt{1011^2+1}} \left(\alpha^{6067} - \beta^{6067} \right) - \frac{1}{2\sqrt{1011^2+1}} \left(\alpha^{2023} - \beta^{2023} \right) \right) \\
&= \frac{1}{2022} \left(C_{6067} - C_{2023} \right) \\
&= \frac{1}{2022} \left(\left(C_{6067} - 1 \right) - \left(C_{2023} - 1 \right) \right) \\
&= \sum_{i=1}^{3033} C_{2i} - \sum_{i=1}^{1011} C_{2i} \\
&= \sum_{i=1012}^{3033} C_{2i}
\end{aligned}$$

So the 2022^{th} child can potentially get the Christmas frog. Assume there exists a positive integer n of at most 4043 and is not 2022 that we can find n Christmas numbers that sum to the intended value. As

$$C_{6067} > \frac{1}{2022} \left(C_{6067} - C_{2023} \right) = \sum_{i=1012}^{3033} C_{2i}$$

So we just need to find non-negative $a_1, a_2, ..., a_{6066}$ such that

$$\sum_{i=1}^{6066} a_i = n \text{ and } \sum_{i=1}^{6066} a_i C_i = \sum_{i=1012}^{3033} C_{2i}$$

Let $b_1, b_2, ..., b_{6066}$ such that $b_i = 1$ if i is an even integer between 2024 and 6066 inclusive, otherwise $b_i = 0$. As $n \neq 2022$, there exists i such that $a_i \neq b_i$. Let i' be the largest such i. It is impossible that $a_{i'} > b_{i'}$ as then

$$\sum_{i=1}^{6066} a_i C_i \geq \sum_{i=i'+1}^{6066} b_i C_i + (b_{i'} + 1)\, C_{i'} \geq \sum_{i=i'}^{6066} b_i C_i + 2022 C_{i'-1} > \sum_{i'=1}^{6066} b_i C_i$$

So $a_{i'} < b_{i'}$, which means i' is an even integer between 2024 and 6066 inclusive and $a_{i'} = 0$.

$$\sum_{i=1}^{6066} a_i C_i = \sum_{i=1}^{6066} b_i C_i$$

$$C_{i'-1} \sum_{i=1}^{i'-1} (a_i - b_i) \geq \sum_{i=1}^{i'-1} (a_i - b_i) C_i = C_{i'} > 2022 C_{i'-1}$$

$$\sum_{i=1}^{i'-1} (a_i - b_i) > 2022$$

$$\sum_{i=1}^{6066} (a_i - b_i) > 2021$$

$$\begin{aligned} \sum_{i=1}^{6066} a_i &> 2021 + \sum_{i=1}^{6066} b_i \\ &= 2021 + 2022 \\ &= 4043 \end{aligned}$$

A contradiction.

$\therefore$ Only the 2022^{th} child can potentially get the Christmas frog.